Orbital Functional Geometry XI

Schrödinger Equation in Orbital Space:
Linearisation, Orbital Potential, and von Mises Spectrum

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Abstract

We apply the Schrödinger equation to the Orbital Space $\mathcal{O}$ of *Orbital Functional Geometry I* (2026). The orbital Hamiltonian is $\hat{H}_{\mathcal{O}} = -(\hbar^2/2m)\Delta_{\mathcal{O}}$, where $\Delta_{\mathcal{O}}$ is the orbital Laplacian acting on $H_{\mathcal{O}} = \{f > 0, \text{ analytic on } S^1\}$ by $\Delta_{\mathcal{O}} f = (\ln f)''$.

The main result is an exact linearisation theorem: the substitution $u = \ln \psi$ transforms the orbital Schrödinger equation into the free Schrödinger equation on u:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta_{\mathcal{O}} \psi \quad \xrightarrow{u = \ln \psi} \quad i\hbar \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 u}{\partial \theta^2}$$

The nonlinearity disappears entirely in the natural coordinates of $\mathcal{O}$.

The stationary states are the von Mises functions $\psi_n^{(R,\phi)}(\theta) = e^{R\cos(n\theta - \phi)}$ with energy levels $E_n = \hbar^2 n^2 / 2m$ and infinite degeneracy parametrised by $(R, \phi) \in \mathbb{R}^+ \times [0, 2\pi)$.

The orbital potential — the effective potential seen by ψ in standard L^2 — is:

$$V_{\mathcal{O}}(\psi, \theta) = \frac{\hbar^2}{2m} \left(\frac{\partial_\theta \psi}{\psi} \right)^2 = \frac{\hbar^2}{2m} (\partial_\theta \ln \psi)^2$$

This is a state-dependent, positive-definite potential with no analogue in linear quantum mechanics.

This construction is distinct from the logarithmic Schrödinger equation (LogSE) of Rosen (1969) and Białynicki-Birula–Mycielski (1976): the LogSE remains nonlinear in all coordinates, while the orbital Schrödinger equation linearises exactly under $u = \ln \psi$.

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Introduction

The Orbital Space $\mathcal{O}$ constructed in Papers I–X carries a natural Hilbert structure $H_{\mathcal{O}}$, a self-adjoint Laplacian $\Delta_{\mathcal{O}}$, and a complete family of eigenfunctions (von Mises functions). These are the ingredients of a quantum mechanical system.

This paper applies the Schrödinger equation in $\mathcal{O}$ and derives its consequences. The main surprise is a complete linearisation: the orbital Schrödinger equation, nonlinear in ψ, becomes the standard free equation in $u = \ln \psi$.

This is not the logarithmic Schrödinger equation of Rosen (1969) and Białynicki-Birula–Mycielski (1976), which remains nonlinear in all coordinate systems and whose solutions are Gaussian (Gaussons). The orbital equation linearises exactly, and its solutions are von Mises functions — exponentials of cosines, not Gaussians.

1 The Orbital Schrödinger Equation

Definition 1 (Orbital Hamiltonian). *The* orbital Hamiltonian *is:*

$$\hat{H}_{\mathcal{O}} = -\frac{\hbar^2}{2m}\Delta_{\mathcal{O}}$$

where the orbital Laplacian acts on $\psi \in H_{\mathcal{O}}$ by:

$$\Delta_{\mathcal{O}}\psi = \frac{d^2}{d\theta^2}\ln\psi = (\ln\psi)''$$

Definition 2 (Orbital Schrödinger equation). *The* orbital Schrödinger equation *is:*

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}_{\mathcal{O}}\psi = -\frac{\hbar^2}{2m}(\ln\psi)''\ \psi$$

This is nonlinear in ψ, since $(\ln\psi)'' = \psi''/\psi - (\psi'/\psi)^2$.

2 Exact Linearisation

Theorem 1 (Linearisation theorem). *The substitution $u = \ln\psi$ transforms the orbital Schrödinger equation into the free Schrödinger equation:*

$$i\hbar\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial\theta^2}$$

The nonlinearity of the orbital equation disappears entirely in the natural coordinates of $\mathcal{O}$.

Proof. Set $u = \ln\psi$, so $\psi = e^u$ and $\partial_t\psi = \dot{u}\,e^u = \dot{u}\,\psi$. The orbital Schrödinger equation becomes:

$$i\hbar\dot{u}\psi = -\frac{\hbar^2}{2m}u''\psi$$

Dividing by $\psi > 0$:

$$i\hbar\dot{u} = -\frac{\hbar^2}{2m}u''$$ □

Remark 1 (Why this differs from the LogSE). *The logarithmic Schrödinger equation (Rosen 1969; Białynicki-Birula–Mycielski 1976) has the form:*

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + b(\ln|\psi|^2)\psi$$

This remains nonlinear under any substitution. Its solutions are Gaussian (Gaussons); it does not linearise. The orbital equation is a different object: it involves $(\ln\psi)''$ rather than $(\ln|\psi|^2)\psi$, and linearises exactly under $u = \ln\psi$.

Remark 2 (The variable $u = \ln\psi$ is the orbital invariant). *The variable $u = \ln\psi$ is precisely the orbital invariant $Q = \ln f/P - b$ of Paper I, restricted to $P = 1$, $b = 0$. The linearisation theorem is therefore not an accident: the natural coordinate of $\mathcal{O}$ is $\ln\psi$, and the dynamics are linear in that coordinate by construction.*

3 Stationary States and Spectrum

Theorem 2 (Stationary states). *The stationary states $\psi(\theta,t) = \psi_n(\theta)e^{-iE_nt/\hbar}$ satisfy $\hat{H}_{\mathcal{O}}\psi_n = E_n\psi_n$, which in the u variable reads $u_n'' = -n^2u_n$.*

The solutions are $u_n(\theta) = R\cos(n\theta - \phi)$, giving:

$$\psi_n^{(R,\phi)}(\theta) = e^{R\cos(n\theta-\phi)}$$

These are the von Mises functions of Paper II.

Theorem 3 (Energy spectrum). *The energy levels are:*

$$E_n = \frac{\hbar^2 n^2}{2m}, \quad n = 0, 1, 2, \ldots$$

with infinite degeneracy at each level $n \geq 1$, parametrised by $(R,\phi) \in \mathbb{R}^+ \times [0, 2\pi)$.

The ground state $n = 0$ has $E_0 = 0$ and $\psi_0 = \text{const}$ (unique up to normalisation).

n	E_n	Eigenfunction	Degeneracy
0	0	$\psi = c$	1
1	$\hbar^2/2m$	$e^{R\cos(\theta-\phi)}$	∞
2	$2\hbar^2/m$	$e^{R\cos(2\theta-\phi)}$	∞
n	$\hbar^2n^2/2m$	$e^{R\cos(n\theta-\phi)}$	∞

Remark 3 (Comparison with the rigid rotor). *The rigid rotor has spectrum $E_n = \hbar^2n^2/2I$ and eigenfunctions $e^{in\theta}$ (complex exponentials), with degeneracy 2 for $n \geq 1$ (states $\pm n$). The orbital spectrum has the same energy levels but different eigenfunctions (von Mises, not complex exponentials) and infinite degeneracy. The two systems have the same spectrum but different Hilbert structures.*

4 Temporal Dynamics

Theorem 4 (General time evolution). *The general state at $t = 0$ decomposes as:*

$$\psi(\theta, 0) = \exp\left(\sum_{n=1}^{\infty} R_n \cos(n\theta - \phi_n)\right)$$

by the decomposition theorem of Paper II. Its time evolution is:

$$\psi(\theta, t) = \exp\left(\sum_{n=1}^{\infty} R_n \cos\left(n\theta - \phi_n - \frac{\hbar n^2}{2m}t\right)\right)$$

Each mode n rotates its phase at angular velocity $\omega_n = \hbar n^2/2m$, quadratic in n.

Corollary 1 (Orbital dispersion). *Two modes n and m with equal initial phase $\phi_n = \phi_m$ dephase at rate $|\omega_n - \omega_m| = \hbar|n^2 - m^2|/2m$. Complete dephasing occurs at time:*

$$T_{nm} = \frac{2\pi m}{\hbar|n^2 - m^2|}$$

The dispersion is quadratic — faster than the linear dispersion of a free wave.

5 The Orbital Potential

Theorem 5 (Orbital potential)**.** *In the standard $L^2(S^1)$ Hilbert space, the orbital Hamiltonian corresponds to the kinetic term plus an effective state-dependent potential. Comparing $\hat{H}_{\mathcal{O}}\psi$ with $(-\hbar^2/2m)\partial_\theta^2\psi + V\psi$:*

$$V_{\mathcal{O}}(\psi, \theta) = \frac{\hbar^2}{2m}\left(\frac{\partial_\theta \psi}{\psi}\right)^2 = \frac{\hbar^2}{2m}(\partial_\theta \ln \psi)^2$$

Proof.

$$-\frac{\hbar^2}{2m}(\ln\psi)''\psi = -\frac{\hbar^2}{2m}\left(\frac{\psi''}{\psi} - \left(\frac{\psi'}{\psi}\right)^2\right)\psi = -\frac{\hbar^2}{2m}\psi'' + \frac{\hbar^2}{2m}\left(\frac{\psi'}{\psi}\right)^2\psi$$

The first term is the free kinetic term; the second is $V_{\mathcal{O}}\psi$. □

Remark 4 (Properties of $V_{\mathcal{O}}$)**.**

- ***Positive definite:*** *$V_{\mathcal{O}} \geq 0$ everywhere, with equality only where ψ is constant*
- ***State-dependent:*** *$V_{\mathcal{O}}$ depends on ψ itself — it is not a fixed external potential*
- ***Gradient energy:*** *$V_{\mathcal{O}}$ measures the squared logarithmic gradient of ψ — the cost of variation in orbital space*
- ***Invariant form:*** *$V_{\mathcal{O}} = \frac{\hbar^2}{2m}(u')^2$ where $u = \ln\psi$ — it is the kinetic energy of u on S^1*

Remark 5 (Relation to Bohmian mechanics)**.** *The Bohmian quantum potential is $Q_{\text{Bohm}} = -(\hbar^2/2m)\nabla^2|\psi|/|\psi|$. The orbital potential $V_{\mathcal{O}} = (\hbar^2/2m)(\nabla \ln\psi)^2$ has a similar structure — both are state-dependent and vanish for uniform ψ — but they are distinct objects. $V_{\mathcal{O}}$ is positive definite; Q_{Bohm} can take any sign. A precise comparison requires working with complex ψ, which is outside the scope of $\mathcal{O}$ as currently defined (where $\psi > 0$).*

6 Energy and Orbital Sobolev Norm

Theorem 6 (Energy as Sobolev norm)**.** *The expectation value of energy in state $\psi = e^u$:*

$$\langle E\rangle = \frac{\hbar^2}{2m}\cdot\frac{1}{2\pi}\int_0^{2\pi}|u'(\theta)|^2\,d\theta = \frac{\hbar^2}{2m}\|u\|^2_{H^1(S^1)}$$

The energy is the H^1 Sobolev norm of $u = \ln\psi$. It is minimised by $u = \text{const}$ (the ground state $\psi = \text{const}$).

Summary of New Results

Result	Statement	Status
Orbital Hamiltonian	$\hat{H}_{\mathcal{O}} = -(\hbar^2/2m)\Delta_{\mathcal{O}}$	Established
Linearisation	$u = \ln\psi$ gives free Schrödinger on u	Established
Distinction from LogSE	LogSE nonlinear; orbital equation linearises	Established
$u = \ln\psi$ is orbital Q	Linearisation follows from geometry of $\mathcal{O}$	Established
Stationary states	Von Mises functions $e^{R\cos(n\theta-\phi)}$	Established
Spectrum	$E_n = \hbar^2 n^2/2m$, infinite degeneracy	Established
vs. rigid rotor	Same spectrum, different eigenfunctions	Established
Time evolution	Phase rotation $\phi_n \to \phi_n + \omega_n t$, $\omega_n \sim n^2$	Established
Quadratic dispersion	Dephasing time $T_{nm} = 2\pi m/\hbar\lvert n^2 - m^2\rvert$	Established
Orbital potential	$V_{\mathcal{O}} = (\hbar^2/2m)(\partial_\theta \ln\psi)^2$	Established
Positive definite	$V_{\mathcal{O}} \geq 0$, state-dependent	Established
Energy as H^1 norm	$\langle E\rangle = (\hbar^2/2m)\lVert u\rVert^2_{H^1}$	Established
Bohm relation	Similar structure, distinct objects	Established
Complex ψ extension	$\mathcal{O}$ currently requires $\psi > 0$	Open
Uncertainty principle	Non-canonical commutator, state-dependent bound	Open

Acknowledgements

This work is the direct continuation of the series *Orbital Functional Geometry* I–X (2026). The linearisation theorem was not anticipated: it emerged from applying the Schrödinger equation to $\mathcal{O}$ and following the substitution $u = \ln\psi$. The recognition that u is the orbital invariant Q of Paper I explained why the linearisation was inevitable. The distinction from the LogSE was verified by literature search before writing.

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References

[1] J. Brindel, *Orbital Functional Geometry I–X*, Preprints, Cotonou, 2026.

[2] G. Rosen, *Dilatation covariance and exact solutions in local relativistic field theories*, Phys. Rev. 183 (1969), 1186–1188.

[3] I. Białynicki-Birula and J. Mycielski, *Nonlinear wave mechanics*, Ann. Phys. 100 (1976), 62–93.

[4] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University Press, 2nd ed., 1986.

[5] D. Bohm, *A suggested interpretation of the quantum theory in terms of hidden variables*, Phys. Rev. 85 (1952), 166–179.